Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00105-3

J. Appl. Maths Mechs, Vol. 66, No. 5, pp. 739–744, 2002
© 2003 Elsevier Science Ltd
All rights reserved. Printed in Great Britain
0021-8928/02/$—see front matter

GLOBALLY CONTROLLABLE SYSTEMS OF RIGID BODIES WITH SEVERAL STABLE REST STATES†

O. R. KAYUMOV

Tara, Omsk region

e-mail: mozhan@rambler.ru

(*Received* 12 *October* 2001)

Natural Lagrangian systems, which model mechanisms with the kinematic structure of a tree and have several stable states of rest, are considered. Sufficient conditions are presented for global controllability, that is, the property that the system can be brought, using an admissible control action, from any initial phase state to any given phase state in a finite time. As an example, global controllability is demonstrated foe an object consisting of several point masses connected in series by rods and sliding without friction on a closed smooth curve in a gravitational field, with a single control force applied in the direction of the velocity to the first link. © 2003 Elsevier Science Ltd. All rights reserved.

1. BASIC DEFINITIONS

In this paper, the controllability properties of dynamical systems are discussed in the language that has emerged in the literature [1, 2]. Continuing previous research [3, 4], we shall consider objects with a cylindrical phase, e.g. multi-link pendulums and the like. All the notation and definitions are taken over from [4] without change.

Natural Lagrangian systems are defined as objects whose Lagrangian, symmetric with respect to time reversal ($t \to -t$), is given by

$$L(\mathbf{q}, \mathbf{q}^\cdot) = \tfrac{1}{2}\mathbf{q}^{\cdot T}\mathbf{A}(\mathbf{q})\mathbf{q}^\cdot - B(\mathbf{q})$$

where $\mathbf{q} = (q_1, \ldots, q_n)^T$ is the configuration vector and $\mathbf{A}(\mathbf{q})$ is the inertia matrix (which is positive-definite). We shall again assume that the scalar potential $\mathbf{B}(\mathbf{q})$ has lower limit: $\mathbf{B}(\mathbf{q}) \geqslant 0$, $\mathbf{B}(0) = 0$, and that the equations of motion have the from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{q}^\cdot}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{u}, \quad \mathbf{u} \in \mathbf{U} \subset \mathbf{R}^n \tag{1.1}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^T$ is the vector of controls, taken from a prescribed bounded set $\mathbf{U}$ containing an interior point $\mathbf{u} = 0$. The values of certain ("angular") coordinates q_i ($i = 1, 2, \ldots, r$) are chosen in the covering space $\mathbf{R}^r \times \mathbf{R}^{n-r}$, corresponding to the configuration space $\mathbf{M} = \mathbf{T}^r \times \mathbf{R}^{n-r}$, where $\mathbf{T}^r$ is an $\mathbf{r}$-dimensional torus. In this situation we shall use the notation $\mathbf{q} \in \mathbf{M}$. A similar definition holds for the phase space $\mathbf{TM} = \mathbf{T}^r \times \mathbf{R}^{2n-r}$, so that $(\mathbf{q}, \mathbf{q}^\cdot) \in \mathbf{TM}$.

If the feedback $\mathbf{u} = \mathbf{u}(\mathbf{q}, \mathbf{q}^\cdot)$ is associated with a separatrix surface in $\mathbf{TM}$, motion over which to a singular point takes an infinite time, then the surface is denoted by $\Omega(\mathrm{u}(\mathbf{q}, \mathbf{q}^\cdot))$. The set of equilibrium positions

$$\zeta = \{(\mathbf{q}, \mathbf{q}^\cdot): \ \mathbf{q}^\cdot = 0, \ \partial B/\partial \mathbf{q} = 0, \ \mathbf{u} = 0\}$$

is non-empty and finite.

For any scalar function $V(\mathbf{y})$ we shall use the following notation: $Qv = \{\mathbf{y}: \partial V/\partial \mathbf{y} = 0\}$ is the set of critical points, $E_V = \{c: c = V(\mathbf{y})\}$ is the set of values, and $H_c(V(\mathbf{y})) = \{\mathbf{y}: V(\mathbf{y}) \leqslant c, c \in E_V\}$ are the domains bounded by the level surfaces of the function.

It is well known [1, 2] that global controllability, say of system (1.1), in the neighbourhood of an equilibrium position need be observed only in the linear approximation, that is, considering the linearized equation $\mathbf{y}^\cdot = \mathbf{G}\mathbf{y} + \mathbf{N}\mathbf{u}$ (of dimension n) and evaluating the controllability matrix $\mathbf{K} = [\mathbf{N}, \mathbf{GN}, \ldots, \mathbf{G}^{n-1}\mathbf{N}]$. The controllability condition is rank $\mathbf{K} = n$ [5], and it is repeated exactly for the system

†*Prikl. Mat. Mekh.* Vol. 66, No. 5, pp. 775–781, 2002.

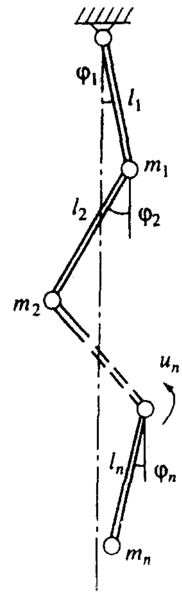

Fig. 1

$\mathbf{y}^{\cdot\cdot} = \mathbf{Gy} + \mathbf{Nu}$, whose order is twice as high (this is precisely the structure of the linearized Lagrange equations).

Estimation of the domain of controllability of non-linear system (1.1) is most interesting in the case when the number of control actions is less that the number of degrees of freedom. As before [4], we shall speak of stabilizability, local controllability or global controllability "with input U_i $(i = 1, \dots, m)$" if the conditions $u_j \equiv 0$ $(j = m+1, m+2, \dots, n)$ are satisfied.

In practical work, one is interested in models (1.1) of mechanical systems which are essentially non-linear. For such systems one can successfully estimate controllability properties in finite domains of the state space only by using the specific features of the classes of objects concerned. Mechanisms of the type of multi-link pendulums have been considered from this point of view [3, 4]. They are characterized by the possibility of stabilizing the whole system in a unique equilibrium position or in a state of steady rotation. In what follows the list of globally controllable object will be enlarged by adding certain systems of rigid bodies that have not one but several stable equilibrium positions.

Sufficient conditions for global controllability have been based [3, 4] on stabilizability properties revealed by Lypunov's direct method in stability theory. In order to extend the well-know Barbashin–Krasovskii theorem [6] to the case of a cylindrical phase space $\mathbf{T}^k \times \mathbf{R}^m$, introduced [7] the concept of a connected Lyapunov function (CLF) $V(\mathbf{y})$ $(\mathbf{y} \in \mathbf{T}^k \times \mathbf{R}^m)$. Such functions are not only positive-definite in Lyapunov's sense $(V(\mathbf{y}) \geqslant 0,\ V(\mathbf{y}) = 0 \Rightarrow \mathbf{y} = 0)$, but also satisfy the condition that all the sets $\mathbf{P} \cap H_c(V(\mathbf{y}))$ are connected in the set $\mathbf{P} \subset \mathbf{T}^k \times \mathbf{R}^m$ under consideration.

For example [7], if the set $\mathbf{Q}_V\backslash 0$ for a Lyapunov function $V(\mathbf{y})$ $(\mathbf{y} \in \mathbf{P})$, defined on a compact manifold $\mathbf{P} \subset \mathbf{T}^k \times \mathbf{R}^m$, consists of a finite number of isolated points, at each of which the matrix $\partial^2 V/\partial \mathbf{y}^2$ is not positive-definite, then $V(\mathbf{y})$ is a CLF on $\mathbf{P}$. In other words, if the function $V(\mathbf{y})$ is not degenerate (i.e. is not a Morse function [8]) on a compact set $\mathbf{P}$ and has only one local minimum (at the point $\mathbf{y} = 0$), then $V(\mathbf{y})$ is a CLF on $\mathbf{P}$.

Example 1. A system of n rigid rods, attached successively by cylindrical hinges, is situated in a vertical plane (Fig. 1). All the angles $\varphi_i (i = 1, 2, \dots, n)$ are measured off by the links of the pendulum from the vertical axis, and the suspension point fixed. The rods are assumed to be weightless, with their masses concentrated at the hinges. Then the potential energy $B(\mathbf{q}) = \Sigma b_i(1 - \cos\varphi_i)$, where $b_i > 0$ $(i = 1, 2, \dots, n)$ is a CLF on $\mathbf{T}^n$.

Example 2. A heavy trolley, on which n pendulums (of different lengths) are suspended in a common plane, is moving horizontally in a straight line (coordinate x), driven by an external horizontal force u satisfying a prescribed bound $|u| \leqslant a$ (Fig. 2). The angles φ_i $(i = 1, 2, \dots, n)$ are measured from the vertical; the configuration vector is $\mathbf{q} = (x, \varphi_1, \dots, \varphi_n)^T \in \mathbf{M} = \mathbf{T}^n \times \mathbf{R}$. The potential $B(\mathbf{q}) = \Sigma d_i(1 - \cos\varphi_i)$, where $d_i > 0$ $(i = 1, 2, \dots, n)$, is not a CLF on $\mathbf{M}$. Nevertheless, at the expense of part

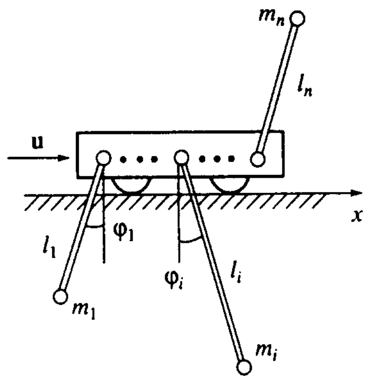

Fig. 2

of the control action ($u = u_1 + u_2$) one can artificially create a "potential well," e.g. by feedback $u_1 = -a(\operatorname{arctg} x)/\pi$. Then the generalized potential

$$\{B(\mathbf{q}) + a[x \operatorname{arctg} x - \tfrac{1}{2}\ln(x^2+1)]/\pi\}$$

will be a CLF on **M**.

One of the results obtained previously [3] will be used below in the following form; for simplicity we will confine out attention to the case of one-dimensional motion.

Proposition 1. Suppose the potential $B(\mathbf{q})$ in system (1.1) is a connected Lyapunov function on **M** and that the sets $H_c(B(\mathbf{q}))$ are compact. Then:

(1) if the system, in free motion ($\mathbf{u} \equiv 0$), does not admit of the particular solution $q_j^{\cdot} \equiv 0$ (excluding equilibrium positions), then it is stabilizable with input u_j ($j = 1, 2, \ldots, n$) on the manifold $\mathbf{TM}\backslash\Omega(u_j)$;

(2) If at the same time system (1.1) is locally controllable in the neighbourhood of all points $(\mathbf{q}_*, 0) \in \zeta$ with the same input u_j, then it is globally controllable when only u_j is applied.

In particular, it follows from Proposition 1 that the n-link pendulum in the vertical plane considered in Example 1 (without friction) is globally controllable by a single bounded torque $|u_n| \leqslant a_n$ applied to the last hinge.

Similarly, one can prove that the system of Example 2 (Fig. 2) is globally controllable on $\mathbf{T}^n \times \mathbf{R}^{n+2}$, provided that the lengths of the pendulums suspended on the trolley are pairwise distinct [3].

2. SYSTEMS WITH A FINITE NUMBER OF STABLE REST STATES

Proposition 1 can be generalized to the case of several local minima of the potential energy $B(\mathbf{q})$, that is, without requiring it to be a connected Lyapunov function over the entire configuration manifold **M**. It will suffice only that all the domains $H_c(B(\mathbf{q}))$ on **M** remain compact; in addition, it is required that the function $B(\mathbf{q})$ should have the properties of a CLF in every connected component of these domains that contains a local minimum. It is assumed under these conditions that the matrices $\partial^2 B/\partial \mathbf{q}^2$ are non-singular at points where $\partial B/\partial \mathbf{q} = 0$. In other words, the potential $B(\mathbf{q})$ must be a Morse function on **M**.

We again require local controllability in the neighbourhoods of all points $(\mathbf{q}_*, 0) \in \zeta$, as well as when there are no particular solutions $q_i^{\cdot} \equiv 0$ (except for rest states) if the controllability is investigated with respect to the input u_i ($i = 1, 2, \ldots, n$).

Since the number of connected components of the sets $H_c(B(\mathbf{q}))$ is finite (for any c), one can reason as shown in the simple of Fig. 3. A heavy point m in a potential well (without friction) may be stopped by a suitable control **u** acting in the direction of the velocity, in one of the states of stable equilibrium. Local controllability in a neighbourhood of either of point r_1 or r_2 makes it possible to be stopped at either of them, without "sticking" at the point r_{12}. In the phase space $(q, q^{\cdot})$ the object can be taken from position $(r_{12}, 0)$ to state $(r_1, 0)$, or equally well to $(r_2, 0)$. Therefore (because of the symmetry of the equations with respect to time reversal) the controllable motion $(r_1, 0) \to (r_{12}, 0) \to (r_2, 0)$ is feasible. As a result, for any states $(q_0, q_0^{\cdot})$ and $(q_k, q_k^{\cdot})$ of the system, the stepwise motion $(q_0, q_0^{\cdot}) \to (r_1, 0) \to (r_2, 0) \to (q_k, q_k^{\cdot})$ (apart from exchanging the roles of r_1 and r_2) is feasible. Thus, the system is globally controllable with input **u** bounded by a quantity as small as desired.

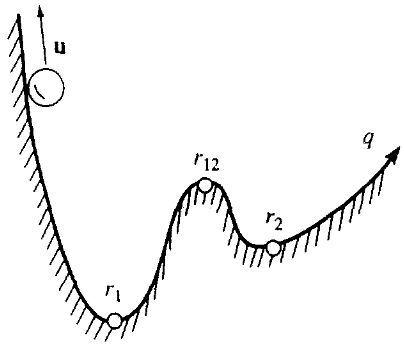

Fig. 3

Proposition 2. Suppose the potential $B(\mathbf{q})$ ($\mathbf{q} \in \mathbf{M}$) for system (1.1) has a finite number of critical points, and that these are non-degenerate. Suppose moreover that a number $c_0 \in E_B$ exists such that the domain $H_c(B(\mathbf{q}))$ is connected and compact for any $c > c_0$.

If in free motion ($\mathbf{u} \equiv 0$) no particular solutions $q_i^{\cdot} \equiv 0$ ($i = 1, 2, \ldots, n$) exist, except for equilibrium positions $(\mathbf{q}_*, 0) \in \zeta$, and in neighbourhoods of these points system (1.1) is locally controllable with the same input u_i, then the system is globally controllable under the action of u_i alone.

Proof. Since the domains $H_c(B(\mathbf{q}))$ are compact for $c > c_0$, it follows that they are compact for any $c \in E_B$, since $H_c \subset H_{c+\varepsilon}$. If the domain $H_c(B(\mathbf{q}))$ is connected for any $c \in E_B$, then $B(\mathbf{q})$ is a CLF and the conditions of Proposition 1 are satisfied.

Suppose (in the general case) that for some $c \leqslant c_0$ the domain $H_c(B(\mathbf{q}))$ was unconnected. Define on $\mathbf{M}$ the set η of critical points $\mathbf{r}_i$ ($i = 1, 2, \ldots, s$) of the potential $B(\mathbf{q})$ at which the matrices $\partial^2 B/\partial \mathbf{q}^2$ are positive-definite. To each point $\mathbf{r}_j \in \eta$ there corresponds a connected component $h_j(c)$ of $H_c(B(\mathbf{q}))$ that contains it, and this component "expands" as c increases.

Since $H_c(B(\mathbf{q}))$ is connected for $c > c_0$, it follows that for any two local minimum points $\mathbf{r}_k$ and $\mathbf{r}_j$ ($k, j = 1, 2, \ldots, s$) (where $B(\mathbf{r}_k) = c_k$, $B(\mathbf{r}_j) = c_j$), a unique number $c_{kj} < c_0$ exists such that

$$h_k(c) \cap h_j(c) = \varnothing \quad \text{for} \quad \max[c_k, c_j] < c < c_{kj}, \quad h_k(c) \cap h_j(c) \neq \varnothing \quad \text{for} \quad c > c_{kj}$$

If there are common points when $c = c_{kj}$, simultaneously for several matrices $h_i(c)$, they may be regarded as element either of $h_k(c_{kj})$ or of $h_j(c_{kj})$. In the set $h_k(c_{kj}) \cap h_j(c_{kj})$, the condition $\partial B/\partial \mathbf{q} = 0$ holds, since in the neighbourhood of that set any two points (of different components of this set – $h_k(c)$ and h_j) do not have codirectional gradients $\partial B/\partial \mathbf{q}$. Since $B(\mathbf{q})$ is a Morse function, the "encounter" of the components h_k and $hj(c)$ as $c \to c_{kj}$ takes placed at a finite number of isolated critical points.

A motion of system (1.1), having begun at one of these points (call it $(\mathbf{r}_{kj}, 0)$) with kinetic energy $T(t_0) = 0$, may be directed into previously selected domain of the configurations ($h_k(c_{kj} - 0)$ or $h_j(c_{kj} - 0)$), due to local controllability. Using the control, we deduce from the condition $dE/dt \leqslant 0$ that $B(t_0) + T(t_0) \geqslant B(t_1) + T(t_1)$ for all $t_1 > t_0$, whence it follows that $B(t_0) \geqslant B(t_1)$, that is, the configuration does not go beyond the bounds of the selected set (e.g. $h_k(c)$).

The absence of particular solutions $q_i^{\cdot} \equiv 0$ ($i \in$ 1, 2, $\ldots$, n) (except for rest states) and local controllability in the neighbourhood of all equilibrium points enables us (according to the logic of Proposition 1) to reach a certain local minimum point $\mathbf{r}_p \in \eta$ ($p = 1, 2, \ldots, s$) in $h_k(c)$.

Inside the set $h_k(c)$, the previous arguments may be repeated for the points $\mathbf{r}_k$ and $\mathbf{r}_p$: find the value of c_{kp}, go from the state $(\mathbf{r}_{kp}, 0)$ to one of the two domains ($h_k(c_{kp} - 0)$ or $h_p(c_{kp} - 0)$), reach a new stable equilibrium, and so on. After a finite number of steps, the system will be in state $(\mathbf{r}_k, 0)$, since the number s is finite. In analogous fashion, starting from a previously encountered "fork" $(r_{kj}, 0)$, one can reach states $(\mathbf{r}_j, 0)$ with a configuration in $h_j(c)$. The symmetry of Eqs (1.1) with respect to time reversal implies that the stepwise passage $(\mathbf{r}_k, 0) \to (\mathbf{r}_{kj}, 0) \to (\mathbf{r}_j, 0)$ is feasible.

Thus, for any two stable rest states, a trajectory of controlled motion connecting them exists. We have at the same time shown that, for any initial conditions $(\mathbf{q}_0, \mathbf{q}_0^{\cdot})$ or $(\mathbf{q}_k, \mathbf{q}_k^{\cdot})$, the object can be stopped in any stable equilibrium position (not known in advance). Because of this same symmetry ($t \to -t$), this implies that the motion $(\mathbf{q}_0, \mathbf{q}_0^{\cdot}) \to (\mathbf{r}_m, 0) \to (\mathbf{r}_l, 0) \to (\mathbf{q}_k, \mathbf{q}_k^{\cdot})$ is feasible (where the subscripts m, $l = 1, 2, \ldots, s$ are to be found according to their meaning). This proves global controllability.

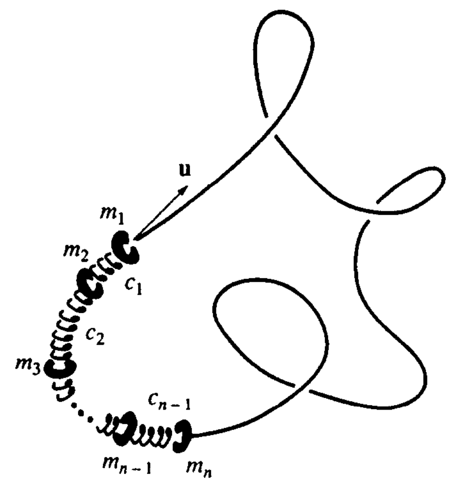

Fig. 4

Example 3. Heavy rings m_i ($i = 1, 2, \ldots, n$) (of negligible size), connected in succession by weightless springs of stiffness c_i ($i = 1, 2, \ldots, n-1$), can slide without friction along a closed wire, without self-intersections, whose curvature at each point is bounded (Fig.4). The only control action is an external force $\mathbf{u}$, $|\mathbf{u}| \leqslant a$, applied at the point m_1 collinear with its velocity. The arc coordinates s_1 of the point m_1 is measured along the wire from a certain fixed position. For the load m_2, the coordinate s_2 is measured from the position of the point corresponding to $s_1 = 0$ and the unstretched spring c_1. Similar conventions apply to the remaining coordinates $s, \ldots, s_n$, so that the deformation of the ith spring will be $(s_{i+1} - s_i)$ ($i = 1, 2, \ldots, n-1$). The configuration vector of the system is $\mathbf{q} = (s_1, s_2, \ldots, s_n)^T \in \mathbf{T}^1 \times \mathbf{R}^{n-1}$.

The Lagrangian is defined by the constant inertia matrix $\mathbf{A}(\mathbf{q}) = \mathbf{A} = \operatorname{diag}(m_i)$ ($i = 1, 2, \ldots, n$) and the potential energy

$$B(\mathbf{q}) = \sum_{i=1}^{n} B_i(s_i) + \frac{1}{2}\sum_{i=1}^{n-1} c_i(s_{i+1} - s_i)^2$$

where $B_i(s_i)$ is the component of the potential of the gravity force for the ith load ($i = 1, 2, \ldots, n$).

In free motion ($u = 0$) the system has no particular solutions $s_1^{\cdot} \equiv 0$ except for rest states. Otherwise, in projection onto an axis tangent to the system at the point m_1 the elastic force and weight would be balanced, that is, we would have constant deformation $(s_2 - s_1)$ of the spring c_1, and subsequently also deformations of all the other springs, corresponding to equilibrium of the object.

Let $(\mathbf{q}_*, 0)$ be any of the rest states. We claim that in its neighbourhood the system is locally controllable with input u. In the linear approximation (substituting $\mathbf{x} = \mathbf{q} - \mathbf{q}_*$) the equation of motion

$$A\mathbf{q}^{\cdot\cdot} = -\partial B/\partial \mathbf{q} + \mathbf{b}u, \quad \mathbf{b} = (1, 0, \ldots, 0)^T$$

becomes

$$\mathbf{A}\mathbf{x}^{\cdot\cdot} = \mathbf{B}_*\mathbf{x} + \mathbf{b}u$$

with $\mathbf{B}_*$ a tridiagonal matrix: the principal diagonal elements are $-(\beta_1 + c_1)$, $-(\beta_2 + c_1 + c_2)$, $\ldots$, $-(\beta_i + c_{i-1} + c_i)$, $\ldots$, $-(\beta_{n-1} + c_{n-2} + c_{n-1})$, $-(\beta_n + c_{n-1})$, those of the diagonals above and below the principal diagonal $c_1, c_2, \ldots, c_{n-1}$, where β_1 ($i = 1, 2, \ldots, n$) are the coefficients of a quadratic form approximating the potential of the gravity force at the point in question.

The substitution

$$\mathbf{A}\mathbf{x} = \mathbf{z}, \quad \mathbf{D} = \mathbf{B}_*\mathbf{A}^{-1}$$

leads to the equation

$$\mathbf{z}^{\cdot\cdot} = \mathbf{D}\mathbf{z} + \mathbf{b}u$$

Applying the rank criterion in its familiar form [1], we obtain a controllability matrix $\mathbf{K} = (\mathbf{b}, \mathbf{D}\mathbf{b}, \ldots, \mathbf{D}^{n-1}\mathbf{b})$ which is upper triangular, and moreover $\det \mathbf{K} = \gamma_1^{n-1}\gamma_2^{n-2} \ldots \gamma_{n-1} \neq 0$, where $\gamma_i = c_i/m_i$ ($i = 1, 2, \ldots, n-1$). Therefore rank $\mathbf{K} = n$, implying local controllability in the neighbourhood of the (arbitrarily) selected rest state.

By Proposition 2, the above system of n heavy rings connected successively by springs along a closed wire turns out to be globally controllable by a single force $\mathbf{u}$, applied to the first ring and bounded by an arbitrarily small quantity a.

Proposition 3. If system (1.1) is globally controllable in $\mathbf{TM} = \mathbf{T}^r \times \mathbf{R}^{2n-r}$, then it is globally controllable in the corresponding covering space $\mathbf{TM}_1 = \mathbf{R}^r \times \mathbf{R}^{2n-r}$.

Proof. It will suffice to show that, considering any two states of rest in $\mathbf{TM}_1$, say $(\mathbf{r}_m, 0)$ and $(\mathbf{r}_l, 0)$, analogous to a state $(\mathbf{q}_1, 0) \in \zeta \subset \mathbf{TM}$ with minimum value of $c_1 = \mathbf{B}(\mathbf{q}_1)$, controllable trajectory connecting them exists. We can then complete the following chain in $\mathbf{TM}_1$.

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0) \to (\mathbf{r}_m, 0) \to (\mathbf{r}_l, 0) \to (\mathbf{q}_k, \dot{\mathbf{q}}_k)$$

in which the first and third passage (in direct and reversed time) are guaranteed by global controllability in $\mathbf{TM}$. At the same time, local controllability is still maintained in the neighbourhood of each state $\mathbf{q}_1, \mathbf{q}_2, \ldots$.

Let us number the potential values $c_i = B(\mathbf{q}_i)$ at the critical points of $\mathbf{M}$ (which are finite in number) in order of increasing $c_1, c_2, \ldots$. In $\mathbf{TM}_1$, consider the set $F_c(B(\mathbf{q})) = \{\mathbf{q} \in \mathbf{R}^n\colon B(\mathbf{q}) < c, c \in [c_1, c_0]\}$, as a function of c. For $c_1 < c < c_2$ the set splits into a denumerable number of connected components, repeating the figure $H_c(\mathrm{B}(\mathbf{q})) \subset \mathbf{M}$ in a "periodic" fashion. Within each component we have one of the infinite analogues $\mathbf{r}_1, \mathbf{r}_2, \ldots$ of the state $\mathbf{q}_1 \in \mathbf{M}$ of the global minimum $B(\mathbf{q})$. By known results [8], when the value $c = c_2$ is reached, some of the components come into contact at points analogous to $\mathbf{q}_2 \in \mathbf{M}$.

Suppose, for example, that the components containing $\mathbf{r}_1$ and $\mathbf{r}_2$ come into contact at a point $\mathbf{r}_{12}$. Then, by the logic of Proposition 2, a controllable motion $(\mathbf{r}_1, 0) \to (\mathbf{r}_{12}, 0) \to (\mathbf{r}_2, 0)$ in $\mathbf{TM}_1$ exists. Repeating similar passages several times, one can connect any rest configurations $\mathbf{r}_s$ and $\mathbf{r}_k$ from an infinite connected "chain" of domains in $F_c(B(\mathbf{q}))$ by a trajectory. When $c = c_3$ the nature of the connectedness of the set $F_c(B(\mathbf{q}))$ changes [8] because some of the "chains" come into contact at points $\mathbf{p}_1, \mathbf{p}_2, \ldots$, analogous to $\mathbf{q}_3 \in \mathbf{M}$. Due to local controllability, one can go from a position $(\mathbf{p}_l, 0) \in \mathbf{TM}_1$ to any of the adjacent components of the set $F_c(B(\mathbf{q}))$. In view of the reversibility of time, this guarantees a motion $(\mathbf{r}_f, 0) \to (\mathbf{p}_l, 0) \to (\mathbf{r}_q, 0)$, where $\mathbf{r}_f$ and $\mathbf{r}_q$ are arbitrary analogues of the point $\mathbf{q}_1 \in \mathbf{M}$, selected in advance from the now connected subset $F_c(B(\mathbf{q}))$.

The argument proceeds by induction on c_i. Each time, within the next, newly connected subset $F_c(B(\mathbf{q}))$, any two analogues of the configuration $\mathbf{q}_1 \in \mathbf{M}$ can be **connected** (indirectly) by a controllable trajectory. If c is sufficiently large, the set $F_c(B(\mathbf{q}))$ will become connected in $\mathbf{R}^n$ (otherwise one would have a contradiction to global controllability in $\mathbf{TM}$). Consequently, any two analogues $\mathbf{r}_m$ and $\mathbf{r}_l$ of the point $\mathbf{q}_1 \in \mathbf{M}$ turn out to be in a common connected domain and can be connected by a controllable trajectory.

Example 4. By Proposition 3, the objects considered previously in Examples 1 and 2 (Figs 1 and 2) are globally controllable in the covering phase space. They may be brought in finite time by a scalar bounded control from any initial state $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ to any prescribed state $(\mathbf{q}_k, \dot{\mathbf{q}}_k)$, stipulating the resulting number of complete revolutions of each separate link.

I wish to thank I. F. Boretskii for suggesting the idea underlying this paper.

REFERENCES

1. KRASOVSKII, N. N., *The Theory of Control of Motion*. Nauka, Moscow, 1968.
2. LEE, E. B. and MARKUS, L., *Foundations of Optimal Control Theory*. Wiley, New York, 1967.
3. KAYUMOV, O. R., The global controllability of certain Lagrangian systems. *Izv. Akad. Nauk SSSR. MTT*, 1986, 6, 16–23.
4. BORETSKII, I. F. and KAYUMOV, O. R., Globally controllable systems of rigid bodies. *Prikl. Mat Mekh*., 1998, **62**, 3, 405–412.
5. KALMAN, R. E., The general theory of control systems. In *Proc. 1st International Conference of the IFAC*. Vol. 2. Izd. Akad. Nauk SSSR, Moscow, 1961, 521–547.
6. BARBASHIN, Ye. A. and KRASOVSKII, N. N., The stability of motion in the large. *Dokl. Akad. Nauk SSSR*, 1952, **86**, 3, 453–456.
7. KAYUMOV, O. R., Asymptotic stability in the large in systems with cylindrical phase space. *Izv. Vuzov. Matematika*, 1987, 10, 61–63.
8. MILNOR, J., *Morse Theory*. Princeton University Press, Princeton, NJ, 1963.

Translated by D.L.